Reply to "Coherent stochastic resonance in the presence of a field"

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We demonstrate that for small deviations from the regimes of its validity, the method of images (MOI) yields results for the passage time statistics that agree very well with the results obtained via direct numerical simulations of the integrate-fire model. The Gitterman-Weiss (GW) [Moshe Gitterman and George H. Weiss, preceding Comment, Phys. Rev. E (to be published)] calculation, while undoubtedly correct, sheds no new light on this problem, since it is valid in the same parameter regime as the MOI. Moreover, the MOI allows one to accurately predict critical behavior observed in the first passage density function.

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For the time-homogeneous problem corresponding to the $q=0$ case of Eq. (5) of [1], the method of images (MOI) yields exact results, as is well known [2]. For $q>0$, however, the MOI is not exact. The fundamental premise in [1] was that for very small signals such that $0<q\ll\mu$, where μ is the constant drift term, the MOI should yield useful approximate results. In fact, the Gitterman-Weiss (GW) approach [3] has precisely the same requirement.

The Bulsara-Lowen-Rees (BLR) work comprises analyses of noise-induced cooperative phenomena and consists of two distinct parts. The first part, based on an application of the MOI, concerns the first passage time density function (FPTDF) and appears in Sec. II of [1]. The second part, in Sec. III, concerns the power spectral density. In Fig. 2 of [1], the FPTDF is shown in the presence and absence of a weak signal for two sets of parameter values. We reiterate that the case for $q=0$ is known to be exact; thus can also be easily verified numerically. In Figs. 1 and 2 we show the FPTDF computed using the MOI (solid curves) and numerical simulations of the dynamics from Eq. (1) of [1] (dashed curves). The parameter values correspond to Fig. 2 of [1] and only the $q>0$ cases are considered. We readily observe the following.

(i) The MOI accurately estimates the location of the peaks and their relative heights. The primary difference between the MOI results and the numerical simulations is a small vertical shift, a quantitative change.

(ii) As μ/q increases, one obtains better agreement between the MOI and the exact numerical simulation results, as expected.

(iii) It is well known [2] that the motion in the BLR model is drift dominated. Noise may change the shape of the FPTDF (its width and tail, for example), but it does not change the mean first passage time (MFPT) $\bar{t}$; for $q=0$ this is given by $t_0=a/\mu$, which is equal to the deterministic passage time. For $0<q\ll\mu$, the MFPT (in the presence of noise) and the deterministic passage time are nearly identical. This passage time was computed directly from the deterministic dynamics using the same perturbation constraint that underlies the MOI in Eq. (9) of [1]. The GW theory is a more precise rendition of Eq. (9) of [1]. As observed by Gitterman and Weiss them-

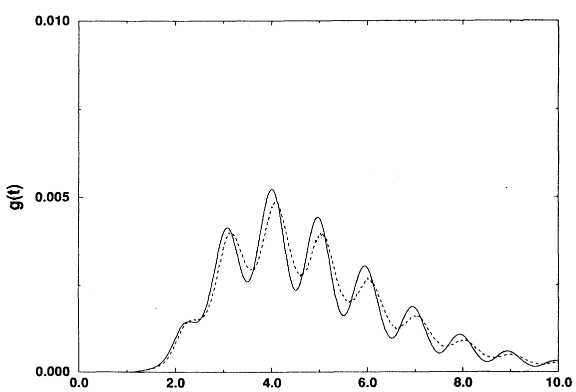

FIG. 1. FPTDF vs normalized time t/T_0 obtained via direct simulation of Eq. (1) of [1] (dashed curve) and the MOI approximation, Eq. (7) of [1] (solid curve). $a=20$, ω $(=2\pi/T_0)=0.1$, $D=0.2$, $\mu=0.065$, and $q=0.03$. See [1] for notation.

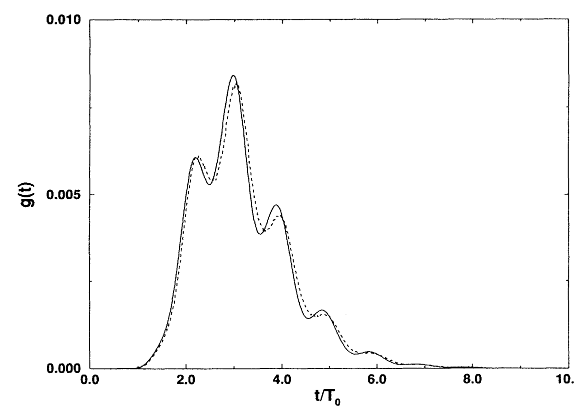

FIG. 2. Same as Fig. 1 with $\mu=0.1$.

selves, the noise might introduce only a small vertical scale shift in the unnormalized MFPT; therefore, whatever multiple frequency resonances may exist in the MFPT should not be strongly noise dependent. We see precisely this kind of effect in the resonance observed in the output signal strength in Figs. 11 and 12 of [1]. This resonance should not be labeled a "stochastic resonance" in the sense of the commonly accepted definition (see, e.g., Refs. [1–6] in [1]); in this regard, the GW terminology is incorrect.

(iv) As observed in [1], the FPTDF for $0<q \ll \mu$ consists of a sequence of peaks superimposed on the FPTDF corresponding to $q=0$. Hence we expect close agreement among the MFPTs computed (i) by numerical integration over the FPTDF, (ii) via the MOI expressions, and (iii) by direct simulations of the underlying stochastic differential equation. To test this assertion we compute the MFPT for the four curves shown in Figs. 1 and 2, for both the $q=0$ case (t_0) and the $q>0$ case ($\bar{t}$), and compare their ratios. For $\mu=0.065$ (Fig. 1), $\bar{t}/t_0=1.0090$ when computed using the numerical simulation (dashed curve) and 0.9999 when computed using the MOI (solid curve). For the $\mu=0.1$ case (Fig. 2), these numbers are 1.0092 and 0.9999, respectively. (All other parameter values correspond to Fig. 2 of [1].) In Fig. 3 we have deliberately violated the perturbation condition for the MOI, choosing $\mu=0.065$ and $q=0.055$ so that $q \ll \mu$ no longer holds. While the features enunciated in item (i) above still hold, we nevertheless obtain excellent agreement between the normalized MFPT corresponding to the numerical simulations (1.0053) and the corresponding quantity obtained via the MOI (0.9894). Hence we expect, although this was not computed in [1], that we will obtain the same resonance behavior that Gitterman and Weiss obtain and, moreover, that the results will accord substantially with numerically computed MFPTs.

In addition to the above observations, we wish to point out that all the resonance behavior depicted in Figs. 3–5 of [1] has been validated by numerical simulation. While there exists a small vertical offset with respect to the curves corresponding to the MOI (related to the shift observed in Figs. 1 and 2 of this paper), the *location* of the peaks in Figs. 3 and 4 of [1] is correctly captured by the MOI, in agreement with our observations in item (iv) above. In recent work [4], we have analyzed the "leaky" integrate-fire model in which the underlying dynamics is an Ornstein-Uhlenbeck process; this is a far richer model, in which the MFPT is strongly noise dependent, and it contains the BLR work as a subset. In this work, we demonstrate the above-described agreement between the MOI and the numerical simulations insofar as the prediction of the noise-induced critical behavior is concerned. Further, we show the existence of a true noise-dependent resonance in the MFPT, in contrast to the BLR and GW works; this is directly attributable to the dependence of the MFPT on noise. In [4] we also describe, in greater mathematical detail, the conditions under which the MOI would be expected to yield acceptable agreement with the exact dynamics. In addition to the abovementioned condition on the signal amplitude, the signal frequency should be very low; strictly speaking, it should be the lowest frequency in the system, corresponding to the often-used adiabatic assumption in modulated stochastic systems. Although the figures in this paper (and in [1]) were obtained with a signal frequency that is somewhat larger than the value predicated on the adiabatic assumption, nevertheless the results of the MOI closely follow those of the numerical simulations.

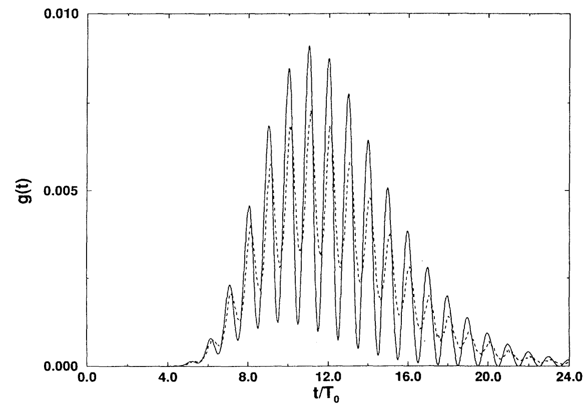

FIG. 3. Same as Fig. 1 with $q=0.055$ and $\omega=0.25$.

In conclusion, we feel that the GW yields no improvement over the MOI since it makes identical assumptions. The MOI theory yields results in good accord with numerical simulation under the appropriate assumptions; it also yields a very good approximation to the FPTDF. In turn, this permits us to explore the above-described resonance behavior (in the FPTDF) and to connect it to the recent definition of stochastic resonance as a synchronization phenomenon [5]. This behavior, described in much greater detail in [4], cannot be described within the GW theory.

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